JOURNAL OF APPROXIMATION THEORY **41**, 87–89 (1984)

NOTE

Distributional Inequalities and van der Corput's Lemma

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Communicated by Oved Shisha

Received September 12, 1982

An approximation method and distributional inequalities are used to generalize a classical lemma due to van der Corput.

INTRODUCTION

The lemma of J. G. van der Corput that is considered here has been fruitfully applied in Fourier analysis and in analytic number theory in estimating trignometric integrals and the Riemann zeta function. In [4], Zygmund states that it is of considerable interest in itself. In this work a distribution theory version of this lemma is obtained which weakens the hypothesis.

NOTATION AND PRELIMINARIES

The space of test functions on the open interval (a, b) is denoted by $\mathscr{D}(a, b)$; its dual space $\mathscr{D}'(a, b)$ is the set of distributions in (a, b). For $f \in L^1_{\text{loc}}(a, b)$ (the space of locally Lebesgue integrable functions on (a, b)), T_f is its associated distribution in $\mathscr{D}'(a, b)$ and $\mathscr{D}^n T_f$ is the nth distributional derivative of f.

Let K be an even nonnegative test function having support $[-1, 1]$ and $\int_{-1}^{1} K(t)\,dt = 1$. Then the regularizations of $f \in L^1_{\text{loc}}(a, b)$ relative to K is the set $\{f_\varepsilon\}$ defined by

$$f_\varepsilon(x) = \int_{-1}^{1} f(x - \varepsilon t)\, K(t)\, dt \qquad \text{for} \quad x \text{ in } (a + \varepsilon, b - \varepsilon).$$

0021-9045/84 $3.00

Distributional Inequalities

The following results and [1] suggest a possible entire theory of distributional inequalities.

Definition 1. For T_1, T_2 in $\mathscr{D}'(a, b)$, $|T_1| \geqslant T_2$ means that $|T_1(\phi)| \geqslant T_2(\phi)$ for all nonnegative ϕ in $\mathscr{D}(a, b)$.

Lemma 2. *Let T_λ be the distribution defined by the constant function $\lambda > 0$, and suppose that $T \in \mathscr{D}'(a, b)$ satisfies $|T| \geqslant T_\lambda$ in (a, b). Then either $T \geqslant T_\lambda$ or $T \leqslant -T_\lambda$ holds exclusively in (a, b).*

Proof. Suppose that there were nonnegative (but not identically zero) test functions ϕ_1 and ϕ_2 in (a, b) such that $T(\phi_1) \geqslant T_\lambda(\phi_1) = \int_a^b \lambda \phi_1(t)\, dt > 0$ and $T(\phi_2) \leqslant -T_\lambda(\phi_2) < 0$. Let $C = -T(\phi_1)/T(\phi_2) > 0$. Then $T(\phi_1 + C\phi_2) = 0$ and $T_\lambda(\phi_1 + C\phi_2) > 0$, since $\phi_1 + C\phi_2$ is a nonnegative test function in (a, b) and is not identically zero. The result now follows by contradiction.

For a real-valued function f, let

$$I(f; a, b) = \left| \int_a^b e^{if(t)}\, dt \right|, \text{ where } (a, b) \text{ may be unbounded.}$$

van der Corput's Lemma (For a proof see [2, p. 264]): Let f be a C^2 convex function on (a, b) and λ a positive constant $\leqslant f''(x)$; then

$$I(f; a, b) \leqslant \frac{8}{\lambda^{1/2}}.$$

Our goal is the following generalized version.

Lemma 3. *Let f be in $L^1_{\text{loc}}(a, b)$ and let λ be a positive constant. If*

$$|\mathscr{D}^2 T_f| \geqslant T_\lambda$$

holds in (a, b) then $I(f; a, b)$ converges, and

$$I(f; a, b) \leqslant \frac{8}{\lambda^{1/2}}.$$

Proof. By Lemma 2 it suffices to consider the case $\mathscr{D}^2 T_f \geqslant T_\lambda$ in (a, b), since otherwise we may replace f by $-f$. Let f_ε be a regularization of f relative to K. For each fixed x in $(a + \varepsilon, b - \varepsilon)$, the function $y \to K((x - y)/\varepsilon)$

has support $[x-\varepsilon, x+\varepsilon] \subset (a, b)$. Hence, it is a nonnegative test function on (a, b). Then it follows that

$$f_\varepsilon''(x) = \frac{1}{\varepsilon^3} \int_{x-\varepsilon}^{x+\varepsilon} f(y) \frac{\partial^2 K((x-y)/\varepsilon)}{\partial x^2} \, dy = \mathscr{D}^2 T_f \left(\frac{1}{\varepsilon} K \left(\frac{x-y}{\varepsilon} \right) \right). \quad (1)$$

Since $\mathscr{D}^2 T_f \geqslant T_\lambda$ in (a, b), we have from (1) that for each x in $(a+\varepsilon, b-\varepsilon)$,

$$f_\varepsilon''(x) \geqslant T_\lambda \left(\frac{1}{\varepsilon} K \left(\frac{x-y}{\varepsilon} \right) \right) = \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \lambda K \left(\frac{x-y}{\varepsilon} \right) dy = \int_{-1}^{1} \lambda K(t) \, dt = \lambda.$$

Thus $f_\varepsilon''(x) \geqslant \lambda$ in $(a+\varepsilon, b-\varepsilon)$ and van der Corput's lemma implies that

$$I(f_\varepsilon \; a+\varepsilon, b-\varepsilon) \leqslant \frac{8}{\lambda^{1/2}}.$$

Because the set of regularizations $\{f_\varepsilon\}$ converge almost everywhere to f in (a, b), we can conclude from Lebesgue's bounded convergence theorem that $I(f; a, b)$ is convergent and $I(f; a, b) = \lim_{\varepsilon \to 0} I(f_\varepsilon; a+\varepsilon, b-\varepsilon) \leqslant 8/\lambda^{1/2}$. This completes the proof.

The following example shows that the distributional inequality condition in Lemma 3 is essential.

EXAMPLE. Let $g(t) = t - L(t)$ and $G(t) = \int_0^t g(x) \, dx$ for $t \in [0, 1]$, where L denotes the Cantor function. Since $\int_0^1 L(t) \, dt = \frac{1}{2}$, we can extend G periodically to a function $\tilde{G}$ on $[0, +\infty)$ as follows:

$$\tilde{G}(t) = G(t-i) \qquad \text{where } i \leqslant t < i+1 \; (i = 0, 1, \ldots,).$$

Then $\tilde{G}'$ is a continuous function of bounded variation on $[0, b]$ for any $b > 0$, and $\tilde{G}''(t) \geqslant 1$ almost everywhere in $(0, +\infty)$. However, since $|\tilde{G}| \leqslant 1$, it is easy to see that $I(G; 0, b)$ diverges to $+\infty$ as $b \to +\infty$.

Remark. It should be pointed out that if f is convex on (a, b) and $f''(t) \geqslant \lambda > 0$ almost everywhere in (a, b) then Lemma 3 is applicable.

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Printed in Belgium